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2004 J. Phys. A: Math. Gen. 37 2693

(http://iopscience.iop.org/0305-4470/37/7/012)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2693–2708 PII: S0305-4470(04)67896-6

Elementary Bäcklund transformations for a discrete Ablowitz–Ladik eigenvalue problem

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Received 22 August 2003, in final form 26 November 2003
Published 4 February 2004
Online at stacks.iop.org/JPhysA/37/2693 (DOI: 10.1088/0305-4470/37/7/012)

Abstract

Elementary Bäcklund transformations (BTs) are described for a discretization of the Zakharov–Shabat eigenvalue problem (a special case of the Ablowitz–Ladik eigenvalue problem). Elementary BTs allow the process of adding bound states to a system (i.e., the add-one-soliton BT) to be 'factorized' to solving two simpler sub-problems. They are used to determine the effect on the scattering data when bound states are added. They are shown to provide a method of calculating discrete solitons—this is achieved by constructing a lattice of intermediate potentials, with the parameters used in the calculation of the lattice simply related to the soliton scattering data. When the potentials, S_n, T_n, in the system are related by $S_n = -T_n$, they enable simple derivations to be obtained of the add-one-soliton BT and the nonlinear superposition formula.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

It is well known that continuous nonlinear evolution equations such as the modified Korteweg–de Vries (mKdV) equation,

$$\frac{\mathrm{d}q}{\mathrm{d}t} + 6q^2\frac{\mathrm{d}q}{\mathrm{d}x} + \frac{\mathrm{d}^3q}{\mathrm{d}x^3} = 0 \tag{1}$$

admit Bäcklund transformations (BTs), which relate one solution $q(x,t)$ to a second solution $\tilde{q}(x,t)$ [1–6]. Defining Q and $\tilde{Q}$ by $q = -\mathrm{d}Q/\mathrm{d}x$ and $\tilde{q} = -\mathrm{d}\tilde{Q}/\mathrm{d}x$, then the BT for the mKdV equation consists of an equation relating the spatial derivatives of Q and $\tilde{Q}$,

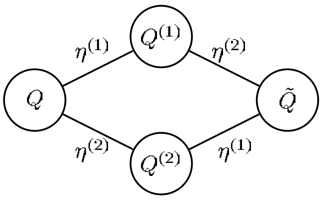
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{Q} + \epsilon Q) = -2\eta\sin(\tilde{Q} - \epsilon Q) \qquad \text{where} \quad \epsilon = \pm 1 \tag{2}$$

and an equation relating the time derivatives of Q and $\tilde{Q}$, which is not relevant here. Equation (2) is known as the 'x-part' of the BT. Here, η is a freely choosable real parameter.

Furthermore, if solution $Q^{(1)}$ is obtained from Q using parameter $\eta = \eta^{(1)}$, and if solution $Q^{(2)}$ is obtained from Q using parameter $\eta = \eta^{(2)}$ then a fourth solution $\tilde{Q}$ can be obtained using the nonlinear superposition formula [7]

$$\tan\frac{\tilde{Q}-Q}{2} = \epsilon\frac{\eta^{(1)}+\eta^{(2)}}{\eta^{(1)}-\eta^{(2)}}\tan\frac{Q^{(1)}-Q^{(2)}}{2}. \qquad (3)$$

This is typically represented by a 'Lamb diagram' [1] of the form



and it provides a method of obtaining 'soliton' solutions of the mKdV equation in closed form [4].

These properties follow from the fact that the mKdV equation is associated with a linear scattering problem (the Zakharov–Shabat eigenvalue problem) [5]

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \begin{pmatrix} -\mathrm{i}\zeta & q(x) \\ r(x) & \mathrm{i}\zeta \end{pmatrix} u \qquad (4)$$

with $r = -q$. Hence other evolution equations with the same underlying scattering problem (the sine-Gordon equation being the most well known) have the x-part of their BTs, and the nonlinear superposition formula, of the same form as the mKdV equation.

Let system (4) have scattering coefficients [5] $a(\zeta), b(\zeta), \bar{a}(\zeta)$ and $\bar{b}(\zeta)$. For $\epsilon = -1$, the x-part of the BT is associated with a change in these scattering coefficients to $\tilde{a}, \tilde{b}, \tilde{\bar{a}}, \tilde{\bar{b}}$, with

$$\tilde{a} = a\frac{\zeta-\zeta_+}{\zeta-\zeta_-} \qquad \tilde{b} = b \qquad \tilde{\bar{a}} = \bar{a}\frac{\zeta-\zeta_-}{\zeta-\zeta_+} \qquad \tilde{\bar{b}} = \bar{b} \qquad (5)$$

where (for $\eta > 0$) $\zeta_+ = \mathrm{i}\eta$ and $\zeta_- = -\mathrm{i}\eta$. The BT therefore adds bound states to the scattering system at $\zeta = \zeta_+$ and $\zeta = \zeta_-$, and is therefore often called an 'add-one-soliton BT'.

When the underlying scattering problem does not have the symmetry $q = -r$, then the x-part of the BT becomes more complex. New potentials $\tilde{q}$ and $\tilde{r}$ are obtained by solving the integro-differential system [8]

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{r}-r) + (\tilde{r}+r)\int_x^\infty \tilde{q}\tilde{r} - qr\,\mathrm{d}x' = 2\mathrm{i}(\zeta_+\tilde{r} - \zeta_- r) \\ &\frac{\mathrm{d}}{\mathrm{d}x}(\tilde{q}-q) + (\tilde{q}+q)\int_x^\infty \tilde{q}\tilde{r} - qr\,\mathrm{d}x' = -2\mathrm{i}(\zeta_-\tilde{q} - \zeta_+ q). \end{aligned} \qquad (6)$$

The scattering coefficients associated with $\tilde{q}$ and $\tilde{r}$ are again related to the scattering coefficients for q and r by equations (5), provided ζ_+ and ζ_- are chosen in the upper and lower halves of the complex plane, respectively.

Remarkably, however, it can be shown [9] that the add-one-soliton BT (6) can be reduced to solving the 'elementary' BTs (EBTs) for intermediate potentials q^- and r^+,

$$\frac{\mathrm{d}r^+}{\mathrm{d}x} - 2\mathrm{i}\zeta_+ r^+ - \frac{\mathrm{i}}{2}r^{+2}q - 2\mathrm{i}r = 0 \qquad (7a)$$

$$\frac{\mathrm{d}q^-}{\mathrm{d}x} + 2\mathrm{i}\zeta_- q^- + \frac{\mathrm{i}}{2}q^{-2}r + 2\mathrm{i}q = 0 \qquad (7b)$$

and then obtaining $\tilde{q}$ and $\tilde{r}$ by the nonlinear superposition formulae

$$\tilde{q} = q + \frac{2(\zeta_- - \zeta_+)}{\frac{2}{q^-} - \frac{r^+}{2}} \qquad \tilde{r} = r + \frac{2(\zeta_+ - \zeta_-)}{\frac{2}{r^+} - \frac{q^-}{2}}. \tag{8}$$

This enables, for example, soliton potentials to be calculated in a purely algebraic way, as was the case when $r = -q$.

This paper obtains the elementary BTs and nonlinear superposition formulae for a discrete version of the continuous scattering problem (4),

$$u_{n+1} = \begin{pmatrix} z & S_n/z \\ T_n z & 1/z \end{pmatrix} u_n \equiv L_n u_n. \tag{9}$$

This system is a special case of the Ablowitz–Ladik eigenvalue problem [10, 11]. It is (or is equivalent to) the underlying linear scattering problem to many nonlinear discrete evolution equations, such as the modified Volterra lattice (mVL) equation [12, 13] (the mVL equation is a discretization of the mKdV equation). It also describes the evolution of two-level systems under the 'hard pulse' approximation [14] and occurs in layer-stripping methods of inverting such systems [15]. Indeed, this provided the initial motivation for this work: the results obtained here enable the simple calculation of discrete versions of 'soliton' (or 'transparent') pulses [16] and truncated soliton pulses [17] for use in selective spin suppression and excitation in magnetic resonance imaging experiments. Other equivalent discretizations exist, as in, for example, [18, 19], but equation (9) is particularly convenient for modelling two-level systems.

From the EBTs for the discrete system, it will be shown how the scattering data are modified after the EBTs are combined to add a soliton to the system. Pure soliton potentials are built up starting with the zero potential ($S_n = T_n = 0$). It will be shown how to calculate these (and their scattering data) in a purely algebraic manner. Finally, the linear system underlying the mVL equation has the symmetry $T_n = -S_n$. Under this symmetry, discrete versions of the add-one-soliton BT (2) and the nonlinear superposition formula (3) are obtained.

2. Discrete elementary Bäcklund transformations

Suppose u_n satisfies the recurrence relation (9),

$$u_{n+1} = \begin{pmatrix} z & S_n/z \\ T_n z & 1/z \end{pmatrix} u_n \equiv L_n u_n. \tag{10}$$

Similar to the continuous case [20], define u_n^+ as

$$u_n^+ = G_n^+ u_n \tag{11}$$

where

$$G_n^+ = \begin{pmatrix} z^2 + \alpha_n^+ & \beta_n^+ \\ \gamma_n^+ z^2 & 1/\nu_+ \end{pmatrix} \tag{12}$$

where ν_+ is a constant that does not depend on n (normally taken equal to 1), and α_n^+, β_n^+ and γ_n^+ are to be chosen so that u_n^+ obeys a recurrence relation of the same form as (9), i.e.,

$$u_{n+1}^+ = \begin{pmatrix} z & S_n^+/z \\ T_n^+ z & 1/z \end{pmatrix} u_n^+ \equiv L_n^+ u_n^+. \tag{13}$$

This requires that [21]

$$L_n^+ G_n^+ = G_{n+1}^+ L_n. \tag{14}$$

In general, G_n^+ becomes singular (i.e., $\det G_n^+ = 0$) at a value of z^2 that depends on n, i.e., $z^2 = -\alpha_n^+ \big/ \big(1 - \nu_+ \beta_n^+ \gamma_n^+\big)$. If it is demanded (in addition to constraint (14)) that G_n^+ be singular at a fixed value of z^2, say z_+^2, then it can be shown that G_n^+ must take the form

$$G_n^+ = \begin{pmatrix} z^2 + z_+^2\big(\nu_+ S_n T_{n-1}^+ - 1\big) & S_n \\ z^2 T_{n-1}^+ & 1/\nu_+ \end{pmatrix} \tag{15}$$

and furthermore that T_n^+ and S_n^+ must satisfy

$$T_{n+1}^+ = \frac{T_n^+ - T_{n+1}/\nu_+}{z_+^2\big(1 - \nu_+ S_{n+1} T_n^+\big)} \tag{16a}$$

$$S_n^+ = \nu_+\big[S_{n+1} + z_+^2 S_n\big(\nu_+ S_{n+1} T_n^+ - 1\big)\big]. \tag{16b}$$

Equation (16*a*) is an EBT for this discrete system: solving it (and equation (16*b*)) maps the old potentials S_n, T_n to the new potentials S_n^+, T_n^+.

A second EBT is obtained by defining $u_n^- = G_n^- u_n$, where

$$G_n^- = \begin{pmatrix} 1/\nu_- & \beta_n^-/z^2 \\ \gamma_n^- & \delta_n^- + 1/z^2 \end{pmatrix} \tag{17}$$

(where ν_- is normally taken equal to -1) and requiring that u_n^- satisfy

$$u_{n+1}^- = \begin{pmatrix} z & S_n^-/z \\ T_n^- z & 1/z \end{pmatrix} u_n^- \equiv L_n^- u_n^-. \tag{18}$$

If G_n^- is also required to be singular at $z^2 = z_-^2$, then it must take the form

$$G_n^- = \begin{pmatrix} 1/\nu_- & S_{n-1}^-/z^2 \\ T_n & (\nu_- T_n S_{n-1}^- - 1)/z_-^2 + 1/z^2 \end{pmatrix} \tag{19}$$

and S_n^-, T_n^- must satisfy

$$S_{n+1}^- = \frac{z_-^2(S_n^- - S_{n+1}/\nu_-)}{1 - \nu_- S_n^- T_{n+1}} \tag{20a}$$

$$T_n^- = \nu_- \left[T_{n+1} + \frac{1}{z_-^2} T_n(\nu_- T_{n+1} S_n^- - 1)\right]. \tag{20b}$$

Equation (20*a*) is the second EBT for this system.

The two EBTs obtained are the discrete versions of equations (7*a*) and (7*b*). The latter equations are obtained in the limit $h \to 0$ after letting

$$\begin{aligned} &S_{n+1} \to hq(x+h) && T_{n+1} \to hr(x+h) \\ &T_{n+j}^+ \to \tfrac{\mathrm{i}}{2\nu_+} r^+(x+jh) && S_{n+j}^- \to -\tfrac{\mathrm{i}}{2\nu_-} q^-(x+jh) \\ &z \to \exp(-\mathrm{i}\zeta h) && z_\pm \to \exp(-\mathrm{i}\zeta_\pm h). \end{aligned} \tag{21}$$

Essentially the same transformations have been derived by others, for example in the context of discrete-time Ablowitz–Ladik equations [19, 22].

Nonlinear superposition formulae for these EBTs can be obtained as for the continuous case [9]. Namely, imagine that BT (16*a*) is used to map potentials S_n, T_n to S_n^+, T_n^+; then BT (20*a*) is used to map S_n^+, T_n^+ to $\tilde{S}_n$, $\tilde{T}_n$ (hence, for example, $\tilde{S}_n$ would be used in place of S_n^-

in equation (20a)). But the same final potentials must be obtained if the BTs are applied in the opposite order, as illustrated by the following diagram:

$$\begin{array}{ccc} S_n^-, T_n^- & \xrightarrow[z_+]{} & \tilde{S}_n, \tilde{T}_n \\ \uparrow z_- & & \uparrow z_- \\ S_n, T_n & \xrightarrow[z_+]{} & S_n^+, T_n^+. \end{array} \tag{22}$$

Then

$$\tilde{G}_n^- G_n^+ = \tilde{G}_n^+ G_n^- \tag{23}$$

where

$$\tilde{G}_n^+ = \begin{pmatrix} z^2 + z_+^2(\nu_+ S_n^- \tilde{T}_{n-1} - 1) & S_n^- \\ \tilde{T}_{n-1} z^2 & 1/\nu_+ \end{pmatrix} \tag{24a}$$

i.e., it equals G_n^+ from equation (15), but with S_n, T_n replaced by S_n^-, T_n^-, and S_n^+, T_n^+ replaced by $\tilde{S}_n, \tilde{T}_n$. Similarly, $\tilde{G}_n^-$ is derived from (19) and equals

$$\tilde{G}_n^- = \begin{pmatrix} 1/\nu_- & \tilde{S}_{n-1}/z^2 \\ T_n^+ & \left(\nu_- T_n^+ \tilde{S}_{n-1} - 1\right)\big/ z_-^2 + 1/z^2 \end{pmatrix}. \tag{24b}$$

Constraint (23) can be shown to require that

$$\tilde{S}_n = \frac{\nu_+}{\nu_-}\left[-z_+^2 S_{n+1} + \left(z_+^2 - z_-^2\right)\frac{S_{n+1} - \nu_- S_n^-}{1 - \nu_+\nu_- T_n^+ S_n^-}\right] \tag{25a}$$

and

$$\tilde{T}_n = \frac{\nu_-}{\nu_+}\left[-\frac{1}{z_-^2} T_{n+1} + \left(\frac{1}{z_-^2} - \frac{1}{z_+^2}\right)\frac{T_{n+1} - \nu_+ T_n^+}{1 - \nu_+\nu_- T_n^+ S_n^-}\right] \tag{25b}$$

i.e., $\tilde{S}_n$ and $\tilde{T}_n$ can be obtained in a purely algrebraic way from the initial potentials S_n, T_n and the intermediate potentials S_n^-, T_n^+.

Equations (25a), (25b) are the nonlinear superposition formulae for the discrete EBTs (16a), (20a). Potentials $\tilde{S}_n$ and $\tilde{T}_n$ are shown in the following section to correspond to adding a soliton (with bound states at $z^2 = z_+^2$ and $z^2 = z_-^2$) to S_n, T_n.

3. Scattering data under discrete Bäcklund transformations

Scattering data for the discrete system (9) are defined as in [10, 11]. Namely (assuming S_n and T_n tend to zero sufficiently fast as $n \to \pm\infty$), let $\phi_n, \bar{\phi}_n, \psi_n$ and $\bar{\psi}_n$ be defined as solutions to (9) with behaviour (for z on the unit circle)

$$\phi_n \to \begin{pmatrix} z^n \\ 0 \end{pmatrix} \qquad \bar{\phi}_n \to \begin{pmatrix} 0 \\ -z^{-n} \end{pmatrix} \qquad \text{as} \quad n \to -\infty \tag{26a}$$

and

$$\psi_n \to \begin{pmatrix} 0 \\ z^{-n} \end{pmatrix} \qquad \bar{\psi}_n \to \begin{pmatrix} z^n \\ 0 \end{pmatrix} \qquad \text{as} \quad n \to \infty. \tag{26b}$$

There can only be two linearly independent solutions, and therefore ψ_n and $\bar{\psi}_n$ must be linearly dependent on ϕ_n and $\bar{\phi}_n$. This is expressed as

$$\phi_n = a(z)\bar{\psi}_n + b(z)\psi_n \qquad \bar{\phi}_n = -\bar{a}(z)\psi_n + \bar{b}(z)\bar{\psi}_n. \tag{27}$$

Functions $a(z), b(z), \bar{a}(z)$ and $\bar{b}(z)$ are the scattering coefficients of the system.

Coefficient $a(z)$ can be analytically extended to outside the unit circle, i.e., to $|z| > 1$. Any zeros of $a(z)$ outside the unit circle are bound states of the system, i.e., discrete eigenvalues of (9). Similarly, $\bar{a}(z)$ can be analytically extended to inside the unit circle, and any zeros of $\bar{a}$ there are bound states. For this system, the scattering coefficients can be shown to be even functions of z. Hence the bound states occur in $\pm$ pairs. The bound states due to the zeros of a will be denoted by $\pm z_{+,j}$, and the bound states due to the zeros of $\bar{a}$ will be denoted by $\pm z_{-,j}$.

At a bound state $z = z_{+,j}$ (or $z = -z_{+,j}$), the solutions ϕ_n and ψ_n are linearly related: $\phi_n = b_j \psi_n$. Similarly, at a bound state $z^2 = z_{-,j}^2$, solutions $\bar{\phi}_n$ and $\bar{\psi}_n$ are linearly related: $\bar{\phi}_n = \bar{b}_j \bar{\psi}_n$. The set $\{a(z), b(z), \bar{a}(z), \bar{b}(z), z_{+,j}, z_{-,j}, b_j, \bar{b}_j\}$ is known as the scattering data of the system.

3.1. The transformed scattering coefficients

Let G_n be defined as

$$G_n = \tilde{G}_n^- G_n^+ = \tilde{G}_n^+ G_n^-. \tag{28}$$

This maps any solution, u_n, of system (9) to a solution, $\tilde{u}_n = G_n u_n$, of the '$\sim$' system

$$\tilde{u}_{n+1} = \begin{pmatrix} z & \tilde{S}_n/z \\ \tilde{T}_n z & 1/z \end{pmatrix} \tilde{u}_n \equiv \tilde{L}_n \tilde{u}_n. \tag{29}$$

Then G_n can be shown to have the form

$$G_n = \begin{pmatrix} \frac{z^2 - z_+^2}{\nu_-} + \frac{(z_-^2 - z_+^2) S_n^- (T_n + \nu_+ z_+^2 T_n^+)}{z_-^2 - \nu_- \nu_+ z_+^2 S_n^- T_n^+} & \frac{S_n}{\nu_-} + \frac{\tilde{S}_{n-1}}{\nu_+ z^2} \\ \frac{T_n}{\nu_+} + \frac{\tilde{T}_{n-1}}{\nu_-} z^2 & \frac{1/z^2 - 1/z_-^2}{\nu_+} + \frac{(1/z_+^2 - 1/z_-^2) T_n^+ (S_n + \nu_- S_n^-/z_-^2)}{1/z_+^2 - \nu_- \nu_+ S_n^- T_n^+/z_-^2} \end{pmatrix}. \tag{30}$$

Suppose $v_n \equiv \binom{v_{1,n}}{v_{2,n}}$ is a solution of the original system (9), with $z^2 = z_+^2$, i.e., $v_{n+1} = L_n(\pm z_+) v_n$. It is useful to note that $-v_{2,n+1} / (\nu_+ z_+^2 v_{1,n+1})$, satisfies the same recurrence relation as T_n^+ in EBT (16*a*). Therefore, T_n^+ can be written as

$$T_n^+ = -\frac{1}{\nu_+ z_+^2} \frac{v_{2,n+1}}{v_{1,n+1}} \tag{31}$$

provided the boundary conditions on v_n are chosen correctly.

Similarly, if w_n satisfies equation (9) with $z^2 = z_-^2$, then S_n^- can be identified with

$$S_n^- = -\frac{z_-^2}{\nu_-} \frac{w_{1,n+1}}{w_{2,n+1}}. \tag{32}$$

Thus there are Darboux transformations (DTs) from solutions v_n and w_n of the original system, to the new potentials T_n^+ and S_n^- (cf [23], where a DT was identified between two alternative forms of the Ablowitz–Ladik system).

Equation (30) can then be rewritten as

$$G_n = \begin{pmatrix} \frac{1}{\nu_-} \left[z^2 - z_+^2 - \frac{\|L_n\| (z_+^2 - z_-^2) w_{1,n+1} v_{2,n}}{z_+ \|vw\|_{n+1}} \right] & \frac{S_n}{\nu_-} + \frac{\tilde{S}_{n-1}}{\nu_+ z^2} \\ \frac{T_n}{\nu_+} + \frac{\tilde{T}_{n-1}}{\nu_-} z^2 & \frac{1}{\nu_+} \left[\frac{1}{z^2} - \frac{1}{z_-^2} - \frac{z_- \|L_n\| \left(\frac{1}{z_-^2} - \frac{1}{z_+^2} \right) w_{1,n} v_{2,n+1}}{\|vw\|_{n+1}} \right] \end{pmatrix} \tag{33}$$

where $\|v\, w\|_{n+1} \equiv v_{1,n+1} w_{2,n+1} - v_{2,n+1} w_{1,n+1}$ and $\|L_n\| = 1 - S_n T_n$.

Assume that $S_n, T_n, \tilde{S}_n$ and $\tilde{T}_n$ all tend to zero as $n \to -\infty$. Then, if $|z_+| > 1$ and $|z_-| < 1$, components $v_{1,n}$ and $w_{2,n}$ both tend to zero (for example, $v_{1,n} \sim z_+^n$, which tends to zero). Also $v_{2,n}/v_{2,n+1} \to z_+$ provided $v_n \neq \phi_n$, and $w_{1,n}/w_{1,n+1} \to 1/z_-$, provided $w_n \neq \bar{\phi}_n$. Then G_n tends to

$$G_n \to \begin{pmatrix} \frac{z^2-z_-^2}{\nu_-} & 0 \\ 0 & \frac{1/z^2-1/z_+^2}{\nu_+} \end{pmatrix} \qquad \text{as} \quad n \to -\infty. \tag{34}$$

Similarly, as $n \to \infty$, components $v_{2,n}$ and $w_{1,n}$ both tend to zero, and $v_{1,n}$ and $w_{2,n}$ both tend to infinity (provided $v_n \neq \psi_n$ and $w_n \neq \bar{\psi}_n$). Then,

$$G_n \to \begin{pmatrix} \frac{z^2-z_+^2}{\nu_-} & 0 \\ 0 & \frac{1/z^2-1/z_-^2}{\nu_+} \end{pmatrix} \qquad \text{as} \quad n \to \infty. \tag{35}$$

Therefore,

$$G_n\phi_n \to \frac{z^2 - z_-^2}{\nu_-} \begin{pmatrix} z^n \\ 0 \end{pmatrix} \qquad \text{as} \quad n \to -\infty. \tag{36}$$

Let $\tilde{\phi}_n$ be a solution to (29) with asymptotic behaviour the same as ϕ_n, i.e.,

$$\tilde{\phi}_n \to \begin{pmatrix} z^n \\ 0 \end{pmatrix} \qquad \text{as} \quad n \to -\infty. \tag{37}$$

From equation (36), $\tilde{\phi}_n$ must equal

$$\tilde{\phi}_n = \frac{\nu_-}{z^2 - z_-^2} G_n\phi_n. \tag{38}$$

Then, from the behaviour of ϕ_n and G_n as $n \to \infty$,

$$\tilde{\phi}_n \to \frac{\nu_-}{z^2 - z_-^2} \begin{pmatrix} \frac{1}{\nu_-}(z^2 - z_+^2) a z^n \\ \frac{1}{\nu_+}(1/z^2 - 1/z_-^2) b z^{-n} \end{pmatrix} \qquad \text{as} \quad n \to \infty \tag{39}$$

and therefore the 'a' and 'b' coefficients for the $\sim$ system are

$$\tilde{a} = \frac{z^2 - z_+^2}{z^2 - z_-^2} a \qquad \tilde{b} = -\frac{\nu_-}{\nu_+} \frac{1}{z^2 z_-^2} b. \tag{40a}$$

Similarly, by considering how $\bar{\phi}_n$ is transformed by G_n, the $\bar{a}$ and $\bar{b}$ coefficients for the $\sim$ system are

$$\tilde{\bar{a}} = \frac{1/z^2 - 1/z_-^2}{1/z^2 - 1/z_+^2} \bar{a} \qquad \tilde{\bar{b}} = -\frac{\nu_+}{\nu_-} z^2 z_+^2 \bar{b}. \tag{40b}$$

Equations (40a) and (40b) tend to the result for the continuous system, equation (5), after the identifications in (21) in the limit $h \to 0$, provided $\nu_- = -\nu_+$ (which is why ν_+ and ν_- are normally taken as 1 and -1).

Therefore system (29) has extra bound states at $z^2 = z_+^2$ and at $z^2 = z_-^2$. This derivation is similar to that obtained for the Schrödinger equation in [24]. The similarity extends to the requirement that v_n cannot be chosen equal to ϕ_n or ψ_n (similarly for w_n) in the Darboux transformation if a bound state is to be added. The need for this constraint is made more apparent when considering the scattering data at the new bound states, as described below.

3.2. The scattering data at new bound states

Coefficient $\tilde{a} = 0$ at $z^2 = z_+^2$, and therefore $\tilde{\phi}_n$ and $\tilde{\psi}_n$ become linearly related, say $\tilde{\phi}_n = b_+\tilde{\psi}_n$, at $z^2 = z_+^2$. To determine b_+, note that

$$[\tilde{\phi}_n \quad \tilde{\psi}_n] = G_n[\phi_n \quad \psi_n]\begin{pmatrix} \frac{\nu_-}{z^2-z_-^2} & 0 \\ 0 & \frac{\nu_+}{1/z^2-1/z_-^2} \end{pmatrix}. \tag{41}$$

This follows from equation (38) and a similarly derived expression for $\tilde{\psi}_n$. Here, $[\phi_n \quad \psi_n]$ means the 2×2 matrix with columns made up of ϕ_n and ψ_n.

The condition $\tilde{\phi}_n = b_+\tilde{\psi}_n$ at $z^2 = z_+^2$ can be written as

$$[\tilde{\phi}_n \quad \tilde{\psi}_n]_{z^2=z_+^2}\begin{pmatrix} 1 \\ -b_+ \end{pmatrix} = 0. \tag{42}$$

Therefore,

$$G_n\big(z^2 = z_+^2\big)[\phi_n \quad \psi_n]_{z^2=z_+^2}\begin{pmatrix} \frac{\nu_-}{z_+^2-z_-^2} & 0 \\ 0 & \frac{\nu_+}{1/z_+^2-1/z_-^2} \end{pmatrix}\begin{pmatrix} 1 \\ -b_+ \end{pmatrix} = 0 \tag{43}$$

and therefore $G_n\big(z^2 = z_+^2\big)$ has kernel

$$\ker G_n\big(z^2 = z_+^2\big) = [\phi_n \quad \psi_n]_{z^2=z_+^2}\begin{pmatrix} \frac{\nu_-}{z_+^2-z_-^2} \\ -\frac{\nu_+ b_+}{1/z_+^2-1/z_-^2} \end{pmatrix}. \tag{44}$$

But (equation (33)),

$$G_n\big(z^2 = z_+^2\big) = \frac{(1 - S_nT_n)\big(z_-^2 - z_+^2\big)}{z_+\|v\ w\|_{n+1}}\begin{pmatrix} \frac{w_{1,n+1}}{\nu_-} \\ -\frac{w_{2,n+1}}{\nu_+ z_-^2} \end{pmatrix}(v_{2,n} \quad -v_{1,n}) \tag{45}$$

and therefore $G_n\big(z^2 = z_+^2\big)$ has kernel

$$\ker G_n\big(z^2 = z_+^2\big) = \begin{pmatrix} v_{1,n} \\ v_{2,n} \end{pmatrix} = v_n. \tag{46}$$

Now v_n could be any solution of (9) (at $z^2 = z_+^2$) other than ϕ_n or ψ_n, and so can be written as

$$v_n = [\phi_n \quad \psi_n]_{z^2=z_+^2}\begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} \tag{47}$$

where α_+ and β_+ are non-zero, but otherwise freely choosable, complex constants.

Finally, comparing equations (44) and (46), β_+/α_+ must equal

$$\beta_+/\alpha_+ = -\frac{\nu_+ b_+}{1\big/z_+^2 - 1/z_-^2}\bigg/\frac{\nu_-}{z_+^2 - z_-^2} \tag{48}$$

and hence b_+ equals

$$b_+ = \frac{\nu_-}{\nu_+}\frac{\beta_+/\alpha_+}{z_+^2 z_-^2}. \tag{49}$$

Note that the requirement that α_+ and β_+ must both be non-zero for bound states to be added makes sense, as b_+ must be both finite and non-zero for a bound state to exist.

The scattering data at $z^2 = z_-^2$ are obtained in the same way. Writing

$$w_n = [\bar{\psi}_n \quad \bar{\phi}_n]_{z^2=z_-^2}\begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} \tag{50}$$

where α_- and β_- are non-zero constants, and noting that

$$G_n(z^2 = z_-^2) = \frac{(1 - S_n T_n)\left(z_-^2 - z_+^2\right)}{z_- \|vw\|_{n+1}} \begin{pmatrix} \frac{v_{1,n+1}}{\nu_-} \\ -\frac{v_{2,n+1}}{\nu_+ z_+^2} \end{pmatrix} (w_{2,n} \quad -w_{1,n}) \tag{51}$$

then $\tilde{\bar{\phi}}_n = \bar{b}_- \tilde{\bar{\psi}}_n$ at $z^2 = z_-^2$, where

$$\bar{b}_- = \frac{\nu_+}{\nu_-} \frac{\alpha_-}{\beta_-} z_+^2 z_-^2. \tag{52}$$

4. Calculation of solitons

Soliton potentials correspond to adding solitons to the 'vacuum', i.e., $S_n = T_n = 0$. The existence of the nonlinear superposition formulae for the elementary BTs allows this to be done very simply.

4.1. The soliton lattice

Let $S_n^{(0,0)} = 0$ and $T_n^{(0,0)} = 0$ be the initial potentials. Let $S_n^{(j,0)}$, $T_n^{(j,0)}$ be mapped to $S_n^{(j+1,0)}$, $T_n^{(j+1,0)}$ by EBT (16*a*) (and (16*b*)) with $z_+ = z_{+,j+1}$. Similarly let $S_n^{(0,k)}$, $T_n^{(0,k)}$ be mapped to $S_n^{(0,k+1)}$, $T_n^{(0,k+1)}$ by EBT (20*a*) (and (20*b*)) with $z_- = z_{-,k+1}$.

Closed-form expressions can be found for these potentials,

$$S_n^{(j,0)} = 0 \tag{53a}$$

$$T_n^{(j,0)} = \frac{1}{\nu_+^j} \sum_{p=1}^{j} \frac{t_p / z_{+,p}^{2n}}{\prod_{q=1, q\neq p}^{j} \left(z_{+,p}^2 - z_{+,q}^2\right)} \tag{53b}$$

$$S_n^{(0,k)} = \frac{1}{\nu_-^k} \sum_{p=1}^{k} \frac{s_p z_{-,p}^{2n}}{\prod_{q=1, q\neq p}^{k} \left(1/z_{-,p}^2 - 1/z_{-,q}^2\right)} \tag{53c}$$

$$T_n^{(0,k)} = 0 \tag{53d}$$

where t_p and s_p are freely choosable complex parameters.

These potentials form the sides of a lattice of potentials $S_n^{(j,k)}$, $T_n^{(j,k)}$, the first few elements of which are

$$\begin{array}{ccccc}
\begin{matrix} S_n^{(0,2)} \\ T_n^{(0,2)} = 0 \end{matrix} & \longrightarrow & \begin{matrix} S_n^{(1,2)} \\ T_n^{(1,2)} \end{matrix} & \longrightarrow & \begin{matrix} S_n^{(2,2)} \\ T_n^{(2,2)} \end{matrix} \\
z_{-,2} \big\uparrow s_2 & \nearrow & \big\uparrow & \nearrow & \big\uparrow \\
\begin{matrix} S_n^{(0,1)} = s_1 z_{-,1}^{2n}/\nu_- \\ T_n^{(0,1)} = 0 \end{matrix} & \longrightarrow & \begin{matrix} S_n^{(1,1)} \\ T_n^{(1,1)} \end{matrix} & \longrightarrow & \begin{matrix} S_n^{(2,1)} \\ T_n^{(2,1)} \end{matrix} \\
z_{-,1} \big\uparrow s_1 & \nearrow & \big\uparrow & \nearrow & \big\uparrow \\
\begin{matrix} S_n^{(0,0)} = 0 \\ T_n^{(0,0)} = 0 \end{matrix} & \xrightarrow[z_{+,1}]{t_1} & \begin{matrix} S_n^{(1,0)} = 0 \\ T_n^{(1,0)} = \frac{t_1}{z_{+,1}^{2n}}/\nu_+ \end{matrix} & \xrightarrow[z_{+,2}]{t_2} & \begin{matrix} S_n^{(2,0)} = 0. \\ T_n^{(2,0)} \end{matrix}
\end{array} \tag{54}$$

The potentials inside the lattice are calculated using the nonlinear superposition formulae, equations (25*a*), (25*b*). Hence,

$$S_n^{(j+1,k+1)} = \frac{\nu_+}{\nu_-}\left[-z_{+,j+1}^2 S_{n+1}^{(j,k)} + \left(z_{+,j+1}^2 - z_{-,k+1}^2\right)\frac{S_{n+1}^{(j,k)} - \nu_- S_n^{(j,k+1)}}{1-\nu_-\nu_+ T_n^{(j+1,k)} S_n^{(j,k+1)}}\right] \quad (55a)$$

$$T_n^{(j+1,k+1)} = \frac{\nu_-}{\nu_+}\left[-\frac{1}{z_{-,k+1}^2} T_{n+1}^{(j,k)} + \left(\frac{1}{z_{-,k+1}^2} - \frac{1}{z_{+,j+1}^2}\right)\frac{T_{n+1}^{(j,k)} - \nu_+ T_n^{(j+1,k)}}{1-\nu_-\nu_+ T_n^{(j+1,k)} S_n^{(j,k+1)}}\right]. \quad (55b)$$

The arrows in lattice (54) indicate which potentials are needed to calculate further elements of the lattice. Soliton potentials correspond to those on the diagonal of the lattice, i.e., $S_n^{(j,j)}$ and $T_n^{(j,j)}$. They can be calculated (in principle) in closed form in terms of the potentials on the edges of the lattice, $T_n^{(j,0)}$ and $S_n^{(0,k)}$.

4.2. Scattering data for solitons

By definition, an N-soliton system with potentials $S_n^{(N,N)}$, $T_n^{(N,N)}$ has bound states at $z^2 = z_{+,j}^2$ and $z^2 = z_{-,j}^2$ for $j = 1, \ldots, N$. Since the scattering coefficients for $S_n = T_n = 0$ are $a(z) = \bar{a}(z) = 1, b(z) = \bar{b}(z) = 0$, the scattering coefficients for the N-soliton system are (equations (40*a*), (40*b*))

$$a(z) = \prod_{j=1}^{N}\frac{z^2 - z_{+,j}^2}{z^2 - z_{-,j}^2} \qquad b(z) = 0 \qquad \bar{a}(z) = \prod_{j=1}^{N}\frac{1/z^2 - 1/z_{-,j}^2}{1/z^2 - 1/z_{+,j}^2} \qquad \bar{b}(z) = 0. \quad (56)$$

To determine the scattering data, b_j and $\bar{b}_j$, at the bound states, $z^2 = z_{+,j}^2$ and $z^2 = z_{-,j}^2$, consider first a one-soliton system $S_n^{(1,1)}$, $T_n^{(1,1)}$. It is calculated from $S_n^{(0,0)} = T_n^{(0,0)} = 0$ with intermediate potentials (equations (53*b*), (53*c*))

$$T_n^{(1,0)} = \frac{1}{\nu_+} t_1 / z_{+,1}^{2n} \qquad S_n^{(0,1)} = \frac{1}{\nu_-} s_1 z_{-,1}^{2n}. \quad (57)$$

But $T_n^{(1,0)}$ can be identified (equation (31)) with

$$T_n^{(1,0)} = -\frac{1}{\nu_+ z_{+,1}^2}\frac{v_{2,n+1}}{v_{1,n+1}} \quad (58)$$

where $v_n \equiv \binom{v_{1,n}}{v_{2,n}}$ is a solution at $z^2 = z_{+,1}^2$ of system (9) under $S_n = T_n = 0$. Equations (57) and (58) can be satisfied by choosing

$$v_n = \begin{pmatrix} z_{+,1}^n / z_{+,1}^2 \\ -t_1 z_{+,1}^2 / z_{+,1}^n \end{pmatrix} \quad (59)$$

and therefore v_n can be written in terms of $\phi_n = \binom{z^n}{0}$ and $\psi_n = \binom{0}{1/z^n}$,

$$v_n = [\phi \quad \psi]_{z^2 = z_{+,1}^2}\begin{pmatrix} 1/z_{+,1}^2 \\ -t_1 z_{+,1}^2 \end{pmatrix}. \quad (60)$$

Hence (equation (47)),

$$\alpha_+ = \frac{1}{z_{+,1}^2} \qquad \beta_+ = -t_1 z_{+,1}^2 \quad (61)$$

and so b at $z^2 = z_{+,1}^2$ for the one-soliton system equals (equation (49)) $-\nu_- t_1 z_{+,1}^2 / \left(\nu_+ z_{-,1}^2\right)$.

The effect of adding further bound states at $z_{\pm,j}$, $j = 2, \ldots, N$ on this b can be calculated from equation (40*a*), with $z^2 = z_{+,1}^2$, and therefore b_1 for the N-soliton system equals

$$b_1 = -\nu_- t_1 z_{+,1}^2 \big/ \left(\nu_+ z_{-,1}^2\right) \times \prod_{k=2}^{N} -\frac{\nu_-}{\nu_+} \frac{1}{z_{+,1}^2 z_{-,k}^2}$$

$$= \left(-\frac{\nu_-}{\nu_+}\right)^N t_1 \frac{1}{z_{+,1}^{2N-4}} \prod_{k=1}^{N} \frac{1}{z_{-,k}^2}. \tag{62}$$

By symmetry, all the b_j are given by

$$b_j = \left(-\frac{\nu_-}{\nu_+}\right)^N t_j \frac{1}{z_{+,j}^{2N-4}} \prod_{k=1}^{N} \frac{1}{z_{-,k}^2}. \tag{63a}$$

Similarly, the $\bar{b}_j$ can be shown to equal

$$\bar{b}_j = -\left(-\frac{\nu_+}{\nu_-}\right)^N s_j z_{-,j}^{2N-4} \prod_{k=1}^{N} z_{+,k}^2. \tag{63b}$$

Hence there is a direct and simple relationship between the soliton scattering data b_j and $\bar{b}_j$, and the parameters, s_j and t_j, used to calculate the soliton potentials in section 4.1.

5. Symmetries and BTs when $S_n = -T_n$

System (9) when $S_n = -T_n$ is an interesting special case as it permits a simple add-one-soliton BT to be constructed, as opposed to requiring two separate EBTs.

When $S_n = -T_n$, then solutions ϕ_n and $\bar{\phi}_n$ are related by

$$\bar{\phi}_n(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_n\left(\frac{1}{z}\right). \tag{64}$$

Similarly,

$$\bar{\psi}_n(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi_n\left(\frac{1}{z}\right). \tag{65}$$

Therefore

$$\bar{a}(z) = a\left(\frac{1}{z}\right) \qquad \bar{b}(z) = b\left(\frac{1}{z}\right) \qquad z_{-,j} = \frac{1}{z_{+,j}} \qquad \bar{b}_j = b_j. \tag{66}$$

If bound states are added at $z^2 = z_+^2$ and $z^2 = z_-^2$, then (equations (40a), (40b)) to preserve the symmetries (66),

$$z_-^2 = \frac{1}{z_+^2} \qquad \text{and} \qquad \nu_- = \pm\nu_+. \tag{67}$$

It can also be shown that the intermediate potentials S_n^- and T_n^+ must be related by

$$S_n^- = -\frac{\nu_+}{\nu_-} T_n^+ = \mp T_n^+. \tag{68}$$

To add these bound states, consider the applying EBTs by the following route:

$$\begin{array}{ccc} & & \tilde{S}_n = -\tilde{T}_n \\ & & \tilde{T}_n \\ & & \uparrow{\scriptstyle z_-} \\ S_n = -T_n, \; T_n & \xrightarrow[z_+]{} & S_n^+, T_n^+ \end{array} \tag{69}$$

where $z_-^2 = 1\big/z_+^2$.

Potential T_n^+ is obtained by the EBT (16*a*). That is,

$$\begin{aligned}\nu_+ T_{n+1}^+ &= \frac{\nu_+ T_n^+ - T_{n+1}}{z_+^2\left(1 - \nu_+ S_{n+1} T_n^+\right)} \\ &= \frac{\nu_+ T_n^+ - T_{n+1}}{z_+^2\left(1 + \nu_+ T_{n+1} T_n^+\right)} \\ &= \frac{1}{z_+^2} \tan\left[\tan^{-1}\left(\nu_+ T_n^+\right) - \tan^{-1} T_{n+1}\right]. \qquad (70)\end{aligned}$$

Potential $\tilde{T}_n$ is obtained using equation (20*b*) with T_n replaced by T_n^+ and S_n^-, T_n^- replaced by $\tilde{S}_n$, $\tilde{T}_n$:

$$\begin{aligned}\tilde{T}_n &= \nu_-\left[T_{n+1}^+ + \frac{1}{z_-^2} T_n^+\left(\nu_- T_{n+1}^+ \tilde{S}_n - 1\right)\right] \\ &= \nu_-\left[T_{n+1}^+ - z_+^2 T_n^+\left(\nu_- T_{n+1}^+ \tilde{T}_n + 1\right)\right]. \qquad (71)\end{aligned}$$

Solving equation (71) for T_n^+ gives

$$\begin{aligned}\nu_- T_n^+ &= \frac{1}{z_+^2} \frac{\nu_- T_{n+1}^+ - \tilde{T}_n}{1 + \nu_- T_{n+1}^+ \tilde{T}_n} \\ &= \frac{1}{z_+^2} \tan\left[\tan^{-1}\left(\nu_- T_{n+1}^+\right) - \tan^{-1} \tilde{T}_n\right]. \qquad (72)\end{aligned}$$

Replacing n by $n+1$ in equation (72) and comparing with equation (70) implies that

$$\frac{1}{\nu_+} \tan\left[\tan^{-1}\left(\nu_+ T_n^+\right) - \tan^{-1} T_{n+1}\right] = \frac{1}{\nu_-} \tan\left[\tan^{-1}\left(\nu_- T_{n+2}^+\right) - \tan^{-1} \tilde{T}_{n+1}\right]. \qquad (73)$$

Bearing in mind that $\nu_- = \pm\nu_+$, this can be written as

$$\tan\left[\tan^{-1}\left(\nu_+ T_n^+\right) - \tan^{-1} T_{n+1}\right] = \tan\left[\tan^{-1}\left(\nu_+ T_{n+2}^+\right) - \epsilon \tan^{-1} \tilde{T}_{n+1}\right] \qquad (74)$$

where

$$\epsilon = \frac{\nu_+}{\nu_-} = \pm 1. \qquad (75)$$

It is then natural to write

$$\nu_+ T_n^+ = \tan y_n^+ \qquad (76a)$$

$$T_n = \tan[y_{n+1} - y_{n-1}] \qquad (76b)$$

$$\tilde{T}_n = \tan[\tilde{y}_{n+1} - \tilde{y}_{n-1}] \qquad (76c)$$

since equation (74) becomes

$$\tan\left[y_n^+ - (y_{n+2} - y_n)\right] = \tan\left[y_{n+2}^+ - \epsilon(\tilde{y}_{n+2} - \tilde{y}_n)\right]. \qquad (77)$$

To solve this, it is sufficient to solve the recurrence

$$y_{n+2}^+ - (\epsilon\tilde{y}_{n+2} - y_{n+2}) = y_n^+ - (\epsilon\tilde{y}_n - y_n) \qquad (78)$$

the solution of which is, without loss of generality,

$$y_n^+ = \epsilon\tilde{y}_n - y_n \qquad (79)$$

(more generally, $y_n^+ = \epsilon\tilde{y}_n - y_n + \lambda$ is a solution, where λ is an arbitrary constant, but λ can be transformed away without changing $\tilde{T}_n$ or T_n).

Hence, equation (72) becomes

$$z_+^2 \tan(\tilde{y}_n - \epsilon y_n) = \tan(\tilde{y}_{n-1} - \epsilon y_{n+1}). \tag{80}$$

This is the add-one-soliton BT under the restriction that it maps $S_n, T_n = -S_n$ to $\tilde{S}_n, \tilde{T}_n = -\tilde{S}_n$. Solving it for given y_n maps pseudopotential y_n to $\tilde{y}_n$, and hence (via equations (76*b*), (76*c*)), T_n to $\tilde{T}_n$. If $z_+^2 = \pm 1$, BT (80) can be solved, giving

$$\tilde{T}_n = \mp \epsilon T_{n+1} \qquad \text{if} \quad z_+^2 = \pm 1. \tag{81}$$

This is not an interesting BT (the two elementary BTs have effectively cancelled themselves out), hence it can be assumed that

$$z_+^2 \neq \pm 1. \tag{82}$$

In the continuous limit, BT (80) becomes BT (2). This follows from the fact that y_{n+j} can be identified with $-Q(x + jh)/2$ and $\tilde{y}_{n+h}$ with $-\tilde{Q}(x + jh)/2$ in the limit $h \to 0$. When $\epsilon = 1$, BT (80) is equivalent to the BT stated in [13] for the mVL equation (identifying z_+^2 here with $\exp(\kappa)$ in [13]).

As for the continuous system, a nonlinear superposition formula exists for BT (80). Suppose this BT maps T_n to $T_n^{(1)}$ (with added bound states given by $z_{+,1}$) and then it maps $T_n^{(1)}$ to $\tilde{T}_n$ (with added bound states given by $z_{+,2}$). Hence,

$$z_{+,1}^2 \tan\left(y_n^{(1)} - \epsilon y_n\right) = \tan\left(y_{n-1}^{(1)} - \epsilon y_{n+1}\right) \tag{83a}$$

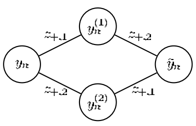
$$z_{+,2}^2 \tan\left(\epsilon \tilde{y}_n - y_n^{(1)}\right) = \tan\left(\epsilon \tilde{y}_{n-1} - y_{n+1}^{(1)}\right). \tag{83b}$$

Here, and later, obvious notation is used for pseudopotentials, e.g., $T_n^{(1)} = \tan\left[y_{n+1}^{(1)} - y_{n-1}^{(1)}\right]$. Alternatively, the BTs can be applied the other way around (map T_n to $T_n^{(2)}$ to $\tilde{T}_n$). Hence,

$$z_{+,2}^2 \tan\left(y_n^{(2)} - \epsilon y_n\right) = \tan\left(y_{n-1}^{(2)} - \epsilon y_{n+1}\right) \tag{83c}$$

$$z_{+,1}^2 \tan\left(\epsilon \tilde{y}_n - y_n^{(2)}\right) = \tan\left(\epsilon \tilde{y}_{n-1} - y_{n+1}^{(2)}\right). \tag{83d}$$

These two choices are represented by a Lamb diagram



(84)

Solving equations (83*b*) and (83*d*) for $\tilde{y}_n$ gives

$$\epsilon \tilde{y}_n = y_n^{(1)} + \tan^{-1}\left[\frac{1}{z_{+,2}^2} \tan\left(\epsilon \tilde{y}_{n-1} - y_{n+1}^{(1)}\right)\right] \tag{85a}$$

$$\epsilon \tilde{y}_n = y_n^{(2)} + \tan^{-1}\left[\frac{1}{z_{+,1}^2} \tan\left(\epsilon \tilde{y}_{n-1} - y_{n+1}^{(2)}\right)\right]. \tag{85b}$$

Hence,

$$y_n^{(2)} - y_n^{(1)} = \tan^{-1}\left[\frac{1}{z_{+,2}^2} \tan\left(\epsilon \tilde{y}_{n-1} - y_{n+1}^{(1)}\right)\right] - \tan^{-1}\left[\frac{1}{z_{+,1}^2} \tan\left(\epsilon \tilde{y}_{n-1} - y_{n+1}^{(2)}\right)\right]. \tag{86}$$

On the other hand,

$$y_n^{(2)} - y_n^{(1)} = \left(y_n^{(2)} - \epsilon y_{n+2}\right) - \left(y_n^{(1)} - \epsilon y_{n+2}\right)$$
$$= \tan^{-1}\left[z_{+,2}^2 \tan\left(y_{n+1}^{(2)} - \epsilon y_{n+1}\right)\right] - \tan^{-1}\left[z_{+,1}^2 \tan\left(y_{n+1}^{(1)} - \epsilon y_{n+1}\right)\right] \tag{87}$$

(using equations (83*a*) and (83*c*)). Therefore, defining

$$x_1 = y_{n+1}^{(1)} - \epsilon y_{n+1} \qquad x_2 = y_{n+1}^{(2)} - \epsilon y_{n+1} \qquad x_3 = \epsilon[\tilde{y}_{n-1} - y_{n+1}] \tag{88}$$

equations (86) and (87) become

$$\tan^{-1}\left(z_{+,2}^2 \tan x_2\right) - \tan^{-1}\left(z_{+,1}^2 \tan x_1\right)$$
$$= \tan^{-1}\left[\frac{1}{z_{+,2}^2}\tan(x_3 - x_1)\right] - \tan^{-1}\left[\frac{1}{z_{+,1}^2}\tan(x_3 - x_2)\right]. \tag{89}$$

Solving equation (89) for $\tan x_3$ gives either

$$\tan x_3 = \frac{-1}{\tan(x_1 + x_2)}$$
$$= \tan\left(x_1 + x_2 + \frac{\pi}{2}\right) \tag{90}$$

or

$$\tan x_3 = \frac{z_{+,1}^2 z_{+,2}^2 - 1}{z_{+,2}^2 - z_{+,1}^2}\left[\frac{z_{+,1}^2 \tan x_1 - z_{+,2}^2 \tan x_2}{1 + z_{+,1}^2 z_{+,2}^2 \tan x_1 \tan x_2}\right]. \tag{91}$$

Case (90) can be neglected: it can be shown to be satisfied when two 'uninteresting' add-one-soliton BTs are combined (with $z_{+,1}^2 = \pm 1$ and $z_{+,2}^2 = \mp 1$).

Case (91) can be written (using equations (83*a*) and (83*c*))

$$\tan x_3 = \frac{z_{+,1}^2 z_{+,2}^2 - 1}{z_{+,2}^2 - z_{+,1}^2}\left[\frac{\tan\left(y_n^{(1)} - \epsilon y_{n+2}\right) - \tan\left(y_n^{(2)} - \epsilon y_{n+2}\right)}{1 + \tan\left(y_n^{(1)} - \epsilon y_{n+2}\right)\tan\left(y_n^{(2)} - \epsilon y_{n+2}\right)}\right]. \tag{92}$$

Hence,

$$\epsilon \tan(\tilde{y}_{n-1} - y_{n+1}) = \frac{z_{+,1}^2 z_{+,2}^2 - 1}{z_{+,2}^2 - z_{+,1}^2}\tan\left(y_n^{(1)} - y_n^{(2)}\right). \tag{93}$$

This can be written in the form

$$\tan(\tilde{y}_{n-1} - y_{n+1}) = \epsilon\frac{a_1 + a_2}{a_1 - a_2}\tan\left(y_n^{(1)} - y_n^{(2)}\right) \tag{94a}$$

where

$$a_1 = \frac{z_{+,1}^2 + 1}{z_{+,1}^2 - 1} \qquad a_2 = \frac{z_{+,2}^2 + 1}{z_{+,2}^2 - 1}. \tag{94b}$$

Equation (93) (or equations (94*a*), (94*b*)) is the nonlinear superposition formula. It allows the potential $\tilde{T}_n$ corresponding to adding two solitons to T_n to be determined algebraically once it is known how to add a single soliton to T_n.

Equation (93) is equivalent to the superposition formula found in [13] for the modified Volterra lattice (mVL) equation. It is useful, however, to obtain these results directly from the linear discrete system, and it is interesting that the elementary BTs provide a straightforward method of doing so.

The superposition formula (as equations (94*a*) and (94*b*)) is also very similar to that found in the continuous case, equation (3). It therefore allows the expression for pulse area for high order soliton potentials obtained in [25] to be used, provided the pulse area of potential T_n (assumed real) is defined as

$$\theta = \sum_{n=-\infty}^{\infty} 2\tan^{-1} T_n. \tag{95}$$

6. Example: one-soliton potentials

It is straightforward to obtain an explicit expression for a one-soliton potential from section 4. This provides a check on the add-one-soliton BT obtained in the previous section.

From lattice (54) and nonlinear superposition formulae (55*a*) and (55*b*), the one-soliton potential is (with $\nu_+ = 1$ and $\nu_- = -1$),

$$\begin{aligned} S_n^{(1,1)} &= \frac{z_{-,1}^2 - z_{+,1}^2}{t_1/z_{+,1}^{2n} - 1/\left(s_1 z_{-,1}^{2n}\right)} \\ T_n^{(1,1)} &= \frac{1/z_{-,1}^2 - 1/z_{+,1}^2}{z_{+,1}^{2n}/t_1 - s_1 z_{-,1}^{2n}}. \end{aligned} \tag{96}$$

Choosing $s_1 = -t_1$ and $z_{-,1} = 1/z_{+,1}$, then $S_n^{(1,1)} = -T_n^{(1,1)}$ and $T_n^{(1,1)}$ can be written as

$$\begin{aligned} T_n^{(1,1)} &= \frac{z_{+,1}^2 - 1/z_{+,1}^2}{z_{+,1}^{2n}/t_1 + t_1/z_{+,1}^{2n}} \\ &= \tan\left[\tan^{-1}\left(-t_1/z_{+,1}^{2(n+1)}\right) - \tan^{-1}\left(-t_1/z_{+,1}^{2(n-1)}\right)\right]. \end{aligned} \tag{97}$$

Comparing this to equation (76*c*), this corresponds to the pseudopotential

$$y_n^{(1,1)} = \tan^{-1}\left(-t_1/z_{+,1}^{2n}\right). \tag{98}$$

Therefore, $y_n^{(1,1)}$ satisfies

$$z_{+,1}^2 \tan y_n^{(1,1)} = \tan y_{n-1}^{(1,1)}. \tag{99}$$

This is the add-one-soliton BT (80), with $y_n = 0$, as expected as $T_n^{(1,1)}$ has been constructed from the zero potential.

Finally, from equation (97), it is easy to verify that (for $t_1 \in \mathbb{R}$, $z_{+,1} \in \mathbb{R}$, $|z_{+,1}| > 1$),

$$\sum_{n=-\infty}^{\infty} 2\tan^{-1} T_n^{(1,1)} = 2\pi t_1/|t_1|. \tag{100}$$

Hence (equation (95)), this is a discrete potential with 'pulse area' 2π.

7. Conclusion

The EBTs (7*a*), (7*b*) for the continuous scattering problem (4) have natural discrete counterparts, equations (16*a*) and (20*a*). By combining the two transformations, nonlinear superposition formulae can be obtained, equations (25*a*) and (25*b*). Potentials $\tilde{S}_n, \tilde{T}_n$ can be calculated such that the system has extra bound states at $z^2 = z_+^2$ and $z^2 = z_-^2$ over the original system with potentials S_n, T_n. Thus, the existence of elementary BTs allows the decomposition of the add-one-soliton BT into two simpler BTs. As pointed out in section 2, equivalent results to these have been found by others.

The main new results of this paper are that firstly, since the intermediate potentials S_n^-, T_n^+ can be expressed in terms of solutions to the original system (i.e., Darboux transformations exist), expressions for the effect of the add-one-soliton BT on the scattering data can be obtained straightforwardly (equations (40*a*), (40*b*), (49) and (52)).

Secondly, solitons can be calculated in a purely algebraic manner via a lattice (54) of intermediate potentials. With initial potentials $S_n = 0$ and $T_n = 0$, successive applications of the EBTs and nonlinear superposition formulae enable the calculation of solitons. The edges of this lattice can be calculated in closed form, and the nonlinear superposition formulae (55a) and (55b) can be used to determine the remainder of the lattice, including the soliton potentials. Explicit expressions (equations (56), (63a) and (63b)) exist for the scattering data of these soliton potentials in terms of the parameters s_j and t_j used to calculate them.

Thirdly, if $S_n = -T_n$, an explicit add-one-soliton BT exists, equation (80), which is most conveniently expressed in terms of pseudopotentials y_n and $\tilde{y}_n$ (equations (76b) and (76c)). A nonlinear superposition formula exists for this BT, equations (94a) and (94b). These results are the discrete versions of those described in the introduction for the mKdV system (equations (2) and (3)). Knowledge of just the form of the one-soliton potential enables calculation of higher order solitons without the lattice needed in the general case when $S_n \neq -T_n$.

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